1.4. Computing interest rate for different period lengths

1.4.1. Relationship between the period length and interest rate

The problem 12 at the end of the previous section was supposed to be a simple, one section bridge, to this section. Here, we elaborate on the subject of relationships between the interest rate and the period length. In order to do that, we will rewrite (1.3) as follows.

$$E(T) = B \times (1+R)^{T} \tag{1.10}$$

where parameter T, unlike the integer value of *n* in (1.3), is a real number.

This notation reflects the fact that the period length can be any real number, and the ending value is a *continuous* function of the period length. The consequences of this substitution are not as trivial as it seems at first. It ties together the interest rate and units of measure for the period. The unit of measure for the period with length *T* is a period of time with one unit length, to which we apply the interest rate. If *R* is the *annual* interest rate, then *T* has to be measured in *years*. Otherwise, the result will be invalid. If some lender applies weekly interest rate of 20%, and lends \$100,000, then the ending value to be repaid in a week is $\$100{,}000 \times (1+0.2)^{1} = \$120{,}000$. However, if an analyst mistakenly measures the period in days, then the calculation produces $\$100{,}000 \times (1+0.2)^{7} = \$429{,}980$. This is, obviously, an invalid result. In fact, it might be a disastrous one. So, we should always remember about this relationship: the interest rate is always associated with the period length.

The lending period has to be measured in units of time the interest rate is applied to.

If this is a weekly interest rate, and the lending period is three weeks, then we should substitute $T = 3$ into equation (1.10). If this is a monthly interest rate, and the month has 30 days, then for the same three weeks period we should substitute $T = (3 \times 7)/30 = 0.7$. If the month has 28 days, then $T = (3 \times 7)/28 = 0.75$.

1.4.2. Computing interest rate for shorter or longer periods. Nominal and effective interest rates

The next question is, how to compute an interest rate for some period, provided we know the interest rate for a period with different length. If the annual interest rate is 120 %, would it be valid to assume that the interest rate for a quarter is (120)/4=30%? What kind of method should we use? This is the area where we have to take a look at the application context.

Do we want to use the obtained interest rate in a compounding scenario and, henceforth, to make use of formula (1.3)? Or, are we going to ignore compounding, and consequently use the non-compounding context and hence formula (1.9)? This is not a hypothetical but, indeed, a practical situation. A financial analyst has to make this choice every day (unless software does this for him). Presently, these issues are allegedly resolved by introduction of certain somewhat artificial constructs. In particular, two important and often used notions such as the *nominal interest rate* and *effective interest rate* are such constructs. In the example above, the *nominal* annual interest rate is 120 %. Then, the monthly interest rate will be calculated as 120/12(months) = 10%. (If thus obtained monthly interest rate should be called a nominal interest rate, or somehow else, depends on the following usage, but presently this question is ignored, in order not to add more ambiguity.)

Anyway, such a monthly interest rate produces the *effective* annual interest rate as follows: $(1+0.1)^{12}=213.8\%$, which is very different from the original 120 %. This happens because of the compounding that is assumed in this case *implicitly*.

Without compounding, the nominal interest rate becomes effective interest rate, which is also equal to 120 %, when we apply similarly the back and forth transformations ($120/12=10\%$, $10\times 12=120\%$).

The approach introducing the notions of the nominal and effective interest rates looks complicated and non-intuitive. However, the proposed explicit introduction of the notion of compounding context helps to clarify the issue, although only to some extent, and the user still has to exercise caution and common sense not be dismayed with the results. According to this approach, in addition to non-intuitive terms we have to add more definitions and certain conditional phrases. Then, the computed value for the effective annual interest rate supposed to become more legitimate. Namely, we have to say exactly this: “Nominal annual interest rate at 120 % compounded monthly”. This phrase still might be a little cryptic, but at least compounding is mentioned this time. This is why financial analysts have to choose the words carefully when explaining what kind of interest rate is discussed with the client. However, the problem is that the general public lacks this refined knowledge, while these people always present on the other side of lending equation.

We agree that this issue of computing interest rate has been overcomplicated without good reasons, which is much explained by historical developments. In fact, there is only one interest rate that can be simply and unambiguously converted to an interest rate for a longer or shorter period. We just have to specify the compounding or non-compounding context. Overwhelming majority of practical applications, such as mortgages and annuities, assume a compounding context.

However, the traditions, conventions and some mentality inertia are things that always should be counted.

Although there are no mathematical or sound business reasons to introduce the nominal and effective interest rates, as well as a conditional wording for their manipulation, we should understand that this is the conventional notation apparatus adapted in this industry and, a least for now, we have to comply with its pitfalls. The problem with these notions is that majority of users do not understand, or quickly forget, these intricacies and simply begin to divide the annual interest by the number of months, if they compute a monthly interest rate from the annual interest rate. When they need semiannual interest rate and they know the monthly interest rate, they multiply the monthly interest rate by six. No reservations, no conditional words, no mentioning of effective or nominal interest rates. This is how the everyday practice corrected these artificial constructs.

1.4.3. Mathematical foundations of interest rate calculations

Let us to consider an example. Suppose we want to do quarterly compounding using correct conventional notions of nominal and effective interest rates. What do we have to do if we know the annual effective interest rate? We should not divide it by four, should we? Should we find a power of $1/4$ of this number? Apparently yes, but we are not sure if the lender would agree with this interpretation given the following consequences.

The annual nominal interest rate is 120 %, as before. The first approach is to use compounding and find a quarterly interest rate as $(1+1.2)^{1/4}-1 \approx 0.2179 = 21.79\%$. With the second approach, when we apply the notions of nominal and effective interest rates, we have to calculate the ending value for a quarter using the original 120%, and applying a simple dividing rule, which produces the interest rate of 30 % for the quarter, effectively creating a non-compounding application context. However, in today's practice, this context is usually not mentioned.

Let us compare the results. The beginning value is $100. In the first case, $E_1 = \$100 \times (1+0.2179) = \$121.79$. The second approach produces $E_1 = \$100 \times (1+0.3) = \$130$. If we do calculation for a one year period, then we have $E_1 = \$100 \times (1+0.2179)^4 \approx \$220$, and $E_1 = \$100 \times (1+0.3)^4 \approx \$285.61$ respectively. The results are substantially different. We have no doubt with regard to the first compounding approach, because we did calculations from scratch according to derived formulas. So, the problem is with the second method.

Mathematical consideration of this phenomenon is as follows. If the second method is true, then the following equality to be held.

$$(1+r)^{\frac{1}{T}} = (1+\frac{r}{T}) \qquad (1.11)$$

The following transformations can be done. We raise both sides of equation (1.11) to power *T*. Both sides are positive. So, this operation is an equivalent mathematical transformation. We obtain:

$$(1+r) = (1+\frac{r}{T})^T \qquad (1.12)$$

The right side of (1.12) is a binomial sequence. We can rewrite it as follows (Salas, 2007).

$$1+r = 1+r+\frac{T(T-1)r^2}{1\cdot 2\cdot T^2}+\frac{T(T-1)(T-2)r^3}{1\cdot 2\cdot 3\cdot T^3}+..... \qquad (1.13)$$

Formula (1.13) shows that the left and right sides of equation are not equal. If *T* is an integer, then the sequence on the right side has a finite number of terms, all of them are positive. If T is not an integer, the number of terms is infinite. So, our assumption is invalid, and, consequently, the method itself is invalid. Nonetheless, this method has a wide acceptance in the industry, and it is used in numerous compounding calculations, while it works *only* for non-compounding scenarios based on formula (1.9) and its variations. So, the conventional approach used in the financial industry is an *approximation*, although overwhelmingly the users do not aware about this specific of the standard approach.

Unfortunately, the idea mixing different contexts in the form of effective and nominal interest rates, and adding conditional phrases to resolve this inherent conflict between different contexts is not exactly efficient. In fact, it is misleading, but this is how the things were arranged. Human mind requires more consistency to avoid conflicts. Maybe at some point people will stop using the nominal interest rate, and begin to use the interest rate as a single notion. If this is the case, then in addition the application context should be defined, which is presently the compounding context almost without exceptions. If this approach is accepted, then recalculation of interest rate to shorter or longer periods becomes simple, straightforward, and unambiguous procedure. However, until that time we should understand and use the current industry methods.

Be aware that in the literature, the interest rate is overwhelmingly used without distinguishing across the boundaries of compounding and non-compounding contexts, as if this is a single territory. This is how people responded to overcomplicated constructs representing the notion of interest rate.

1.4.4. Computing interest rates. Numerical examples

Below, we provide a numerical example for a smaller value of interest rate, in order to see, how critical is the mixture of different contexts in this case. Let us assume $r = 5\%$. Then, the compounding approach delivers the result

$$E = \$100 \times (1 + 0.05)^{1/4} = 101.227$$

The second, proportional or non-compounding approach produces

$$E = \$100 \times (1 + 0.0125) = 101.25$$

The difference is 1.3 cents, which doesn't look as a big variation. The problem is that this is a *systematic* error. If we use invalid interest rate to calculate the ending value for four periods in the compounding scenario, then we will get the difference of 9.5 cents on a sum of $100. Eight periods produce 20 cents. In practice, the number of periods can be tens and hundreds, which is a common situation with mortgages, and transaction values much bigger than our $100. So, this can be noticeable amount when it is accumulated across multiple periods and/or across multiple financial instruments. The error grows rapidly with the increase of interest rate, far quicker than linear dependencies. This fact is illustrated by Table 1.3 and Fig. 1.2.

Table 1.3. Differences in interests produced by two methods for the same principal.

Interest Rate, %	Interest, $ difference for one period per $100, between the compounding and nominal interest rate methods	Difference, % accrued for the total period, composed of four periods	Difference, % accrued for eight smaller periods
5	0.023	0.095	0.2
50	1.83	10.18	31.6
80	4.18	27.36	106.0
120	8.21	65.61	331,7

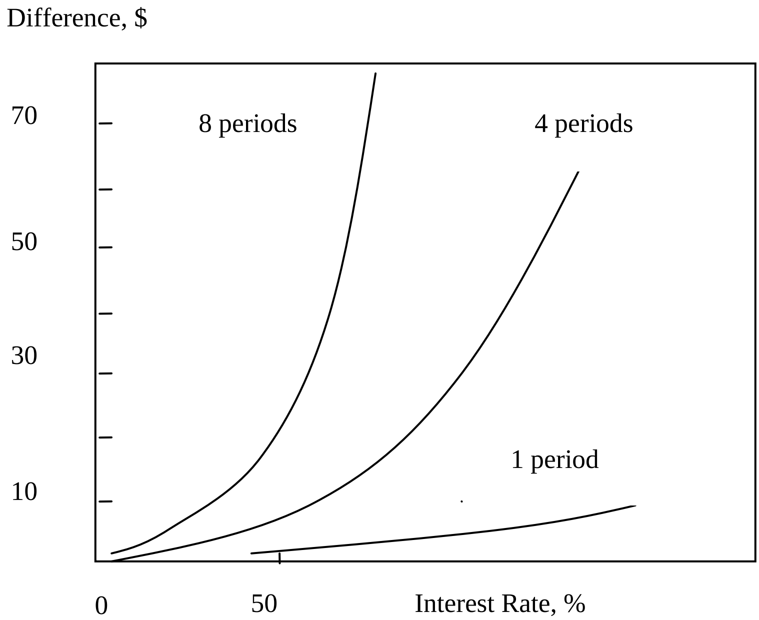

Fig. 1.2. Difference between the interests computed by two methods.

In the first column of the Table 1.3, the first method assumes a compounding context, while the second method uses an effective interest rate and a nominal interest rate for computing interest rate for smaller periods.

So, analysts have to exercise consistent approach when manipulating interest rates. This consistency presumes to remain within the boundaries of compounding or non-compounding contexts, and not to cross the border line between them. Otherwise, the results will be invalid.

1.4.5. Problems and Exercises:

1. The interest rate is applied to a period of 200 days. What value of T in formula (1.10) has to be used if we calculate the ending value for the following periods: 120 days; 2.24 days; 2 weeks; half a year (the year has 366 days); 72 hours; 3 days and 3 hours; two months (March and April).

2. An annual nominal interest rate is 24 %. What is the effective annual interest rate compounded monthly and semimonthly? (Hint: consider the use of formula 1.10.)

3. A nominal interest rate for decade is 0.012. What would be the effective annual interest rate compounded monthly? (Although the decade long period is unusual, the periods can have any length. One year period is a matter of convenience, but not a mathematical or business restriction.)

4. An effective annual interest rate is 0.1268. What is the annual nominal interest rate?

5. What interest is more beneficial to quote for the lender, nominal or effective? Assume that the number is the same.

6. Is the result from the problem 5 always held true? If this is not true, explain why. (Hint: recall the discussion about the properties of compounding.)

7. The borrower is quoted an annual nominal interest rate at 0.6. He wants to borrow $100,000 for a period one year. Using this loan, he will earn $175,000. Is it a good deal for him if the interest is computed using the effective interest rate compounded monthly?

8. The borrower is quoted an annual interest rate at 9 % for a two years loan. The interest and principal to be paid all at once at the end of the second year. The borrower did not ask any questions and signed the loan agreement. What could be the final amount to be paid? Provide two possible scenarios assuming that the lender is also aware about the existence of nominal and effective interest rates.

9. Some credit cards have the terms such as “24 % interest rate compounded weekly”. Let us assume that somebody bought a car at $30,000. He pays interest once a year. How much interest he will pay for ten years?

10. The borrowed amount was $2000. The interest paid after one year was $300. What was the annual nominal interest rate if the interest was calculated based on annual effective interest rate compounded monthly?

11. An annual nominal interest rate is 0.07. What is the interest to be paid in three years and a half on the principal amount $2,300,000.

12. The interest and principal paid after 4.5 years is $1200. The principal was $9000. What was the annual nominal interest rate?

1.5. Continuous compounding.

The previous section hopefully convinced us that we should know the context of the problem, which is whether compounding or non-compounding. There is a possibility to merge them together, but this has to be done cautiously and rightly, and we will present such a case later. An attempt to mix these concepts through the nominal and effective interest rates has inconsistencies. However, it is still in use and the reader should understand its specific. In this section, we will enhance the understanding of compounding context thoroughly studying its properties.

Previously, we considered discrete periods only. However, compounding is a very beneficial option for the lender allowing to receive higher interest while quoting the same interest rate for a period. From the lender perspective, it makes sense to have as many periods as possible, ideally infinitely small and countless. Mathematics provides appropriate quantitative instruments.